BOTT PERIODICITY AND STABLE QUANTUM CLASSES

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Abstract. We use Bott periodicity to relate previously defined quantum classes to certain "exotic Chern classes" on BU. This provides an interesting computational and theoretical framework for some Gromov-Witten invariants connected with cohomological field theories. This framework has applications to study of higher dimensional aspects of Hofer geometry of $\mathbb{CP}^n$, one of which we discuss here. The form of the main theorem, invites us to also speculate here on some "quantum" variants of Mumford's original conjecture (Madsen-Weiss theorem) on the cohomology of stable moduli space of Riemann surfaces.

1. Introduction

In this paper we continue to investigate the theme of topology via Gromov-Witten theory and vice-versa. The topology in question is of certain important infinite dimensional spaces. Here the main such space is BU, and we show that Gromov-Witten theory (in at the moment mysterious way) completely detects its rational cohomology. The first application of this is for the study of higher dimensional aspects of Hofer geometry of $\mathrm{Ham}(\mathbb{CP}^n, \omega_{st})$, and we are able to prove a certain rigidity result for the the embedding $SU(n) \to \mathrm{Ham}(\mathbb{CP}^{n-1})$. Another topological application is given in [13], where we use the main result of this paper to probe topology of the configuration space of stable maps in BU, which might be the first investigation of this kind.

More intrinsically, we get some new insights into Gromov-Witten invariants themselves, as through this "topological coupling" and some transcendental (as opposed to algebraic geometric in nature) methods we will compute some rather impossible looking Gromov-Witten invariants. These methods involve Bott periodicity theorem and differential geometry on loop groups.

Lastly, this paper is a conceptual setup for some interesting "quantum" variants of Mumford's original conjecture on cohomology of the stabilized moduli space of Riemann surfaces, and this further fits into the above mentioned theme of topology via Gromov-Witten theory.

1.1. **Outline.**

One of the most important theorems in topology is the Bott periodicity theorem, which is equivalent to the statement:

$$BU \simeq \Omega SU, \tag{1.1}$$

where SU is the infinite special unitary group. On the space BU we have Chern classes uniquely characterized by a set of axioms. It turns out that the space ΩSU also has natural *intrinsic* cohomology classes, characterized by axioms, but with a somewhat esoteric coefficient ring: $\widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$, the unital completion of (ungraded) formal quantum homology ring $QH(\mathbb{CP}^\infty)$, which is just a free polynomial algebra over $\mathbb{C}$ on one generator. Here is an indication of how it works for $\Omega SU(n)$. Consider

the Hamiltonian action of the group $SU(n)$ on $\mathbb{CP}^{\text{n-1}}$. Using this, to a cycle $f : B \to \Omega SU(n)$ we may associate a $\mathbb{CP}^{\text{n-1}}$-bundle P_f over $B \times S^2$, with structure group $SU(n)$, by using f as a clutching map. Gromov-Witten invariants in the bundle P_f induce cohomology classes

$$qc_k(P_f) \in H^{2k}(B, QH(\mathbb{CP}^{\text{n-1}})). \tag{1.2}$$

(More technically, we are talking about parametric Gromov-Witten invariants. The bundles P_f always have a naturally defined deformation class of fiber-wise families of symplectic forms. Although very often the total space turns out to be Kahler in which case one can really talk about Gromov-Witten invariants and the discussion coincides.) These cohomology classes have analogues of Whitney Sum and naturality axioms as for example Chern classes. There is a also a partial normalization.

We will show that these classes stabilize and induce cohomology classes on $\Omega SU \simeq BU$. These stable quantum classes satisfy a full normalization axiom. However, with that natural normalization stable classes fail the dimension property i.e. $qc_k(E)$ does not need to vanish if E is stabilization of rank r bundle with $r < 2k$. This is somehow reminiscent of phenomenon of generalized (co)-homology. Thus we may think of these classes as *exotic Chern classes*.

Theorem 1.1. *The induced classes qc_k on $\Omega SU \simeq BU$ are algebraically independent and generate the cohomology with the coefficient ring $\widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$.*

Stabilization in this context is somewhat analogous to semi-classical approximation in physics. The "fully quantum objects" are the classes $qc_k \in \Omega SU(n)$, and they are what's important in geometric applications, for example in Hofer geometry. We are still far from completely computing these classes, but as a corollary of the above we have the following.

Theorem 1.2. *The classes qc_k on $\Omega SU(n)$ are algebraically independent and generate cohomology in the stable range $2k \leq 2n-2$, with coefficients in $QH(\mathbb{CP}^{n-1})$.*

Since quantum classes on $\Omega SU(n)$ are pulled back from classes on $\Omega \mathrm{Ham}(\mathbb{CP}^{n-1})$ as a simple topological corollary of Theorem 1.1 we obtain another proof of:

Theorem 1.3 (Reznikov, [10]). *The natural inclusion $\Omega SU(n) \to \Omega Ham(\mathbb{CP}^{n-1})$ is injective on rational homology in degree up to $2n-2$.*

Reznikov's argument is very different and more elementary in nature, he also proved a stronger topological claim, (he did not have conditions on degree) but of course his emphasis was different.

1.2. **Applications to Hofer geometry.** For a closed symplectic manifold (M, ω) recall that the (positive) Hofer length functional $L^+ : \mathcal{L}\mathrm{Ham}(M^n, \omega) \to \mathbb{R}$ is defined by

$$L^+(\gamma) := \int_0^1 \max(H_t^\gamma) dt, \tag{1.3}$$

where H_t^γ is a time dependent generating Hamiltonian function for γ normalized by the condition

$$\int_M H_t^\gamma \omega^n = 0.$$

Let

$$j^E : \Omega^E \mathrm{Ham}(M, \omega) \to \Omega \mathrm{Ham}(M, \omega)$$

denote the inclusion where $\Omega^E \mathrm{Ham}(M,\omega)$ is the sub-level set with respect to the functional L^+. Define

$$(\rho, L^+) : H_*(\Omega \mathrm{Ham}(\mathbb{CP}^{n-1})) \to \mathbb{R},$$

to be the function

$$(\rho, L^+)(a) = \inf\{E| \text{ s.t. } a \in \mathrm{image}\, j^E_* \subset H_*(\Omega \mathrm{Ham}(\mathbb{CP}^{n-1}))\}. \tag{1.4}$$

Let us also denote by i the natural map

$$i : \Omega SU(n) \to \Omega \mathrm{Ham}(\mathbb{CP}^{n-1}),$$

and by i^*L^+ the pullback of the function L^+. We have an analogously defined function

$$(\rho, i^*L^+) : H_*(\Omega SU(n)) \to \mathbb{R}.$$

Theorem 1.4. *If $a \neq 0 \in H_{2k}(\Omega SU(n))$ then $(\rho, L^+)(i_*a) = 1$, provided that $2 \leq 2k \leq 2n-2$, where the standard symplectic form on $\mathbb{CP}^{n-1}$ is normalized by the condition that the symplectic area of a complex line is 1. Moreover we have the following Hamiltonian rigidity phenomenon:*

$$(\rho, i^*L^+)(a) = (\rho, L^+)(i_*a) = 1, \tag{1.5}$$

for class a satisfying same conditions.

What is already interesting is that $(\rho, L^+)(i_*a) \neq 0$, as $\mathrm{Ham}(\mathbb{CP}^{n-1})$ is a very complicated infinite dimensional metric space, and sublevel sets $\Omega^E \mathrm{Ham}(\mathbb{CP}^{n-1})$ may have interesting homology for arbitrarily small E.

It's worth noting that I do not know if this theorem remain true for the full Hofer length functional, obtained by integrating the full oscillation $\max H^\gamma_t - \min H^\gamma_t$. Our argument does break down in this case, as it is not clear how to simultaneously bound both L^+ and L^-.

Excursion: On "quantization" of Mumford conjecture. The discussion in this section is strictly speaking outside the scope of methods and main setup of this paper. Nevertheless it may be helpful to some readers in order to put Theorem 1.1 into some further perspective. Let Σ be a closed Riemann surface of genus g, with a distinguished point x_∞. Let $\mathcal{I}_{g,n}$ be the set of isomorphism classes of tuples (E, α, j), where E is a holomorphic $SL_n(\mathbb{C})$-bundle and α an identification of the fiber over x_∞ with $\mathbb{C}^n$ and j a complex structure on Σ. It follows from [9, Section 8.11] that $\mathcal{I}_{g,n}$ has a natural structure of an infinite dimensional manifold.

Example 1.5. *Take $\Sigma = S^2$, the corresponding space $\mathcal{I}_{0,n}$, is naturally diffeomorphic to $\Omega SU(n)$. This fact is related to the Grothendieck splitting theorem for holomorphic vector bundles on S^2, or more appropriately in this context known as Birkhoff factorization, see* [9, Section 8.10].

On $\mathcal{I}_{g,n}$ there are natural characteristic cohomology classes qc_k defined analogously to classes qc_k on $\Omega SU(n)$. These classes are defined by counting certain holomorphic sections of the projectivization of the bundles (E, α, j), so that in particular we are now talking about genus g Gromov-Witten invariants.

Remark 1.6. *Note that the holomorphic structure will almost never be regular and must be perturbed within the class of* π-compatible *complex structures, see Definition 2.2. For computations such perturbations maybe unfeasible due to very complex and singular moduli spaces that arise, as apparent already in the proof of*

Proposition 2.4, in the genus 0 case. Instead it is preferable to work with virtual moduli cycle and virtual localization, for some naturally occurring circle actions, e.g. there is a natural Hamiltonian circle action on the Kahler manifold $\mathcal{I}_{0,n} = \Omega SU(n)$ induced by the energy functional $E : \Omega SU(n) \to \mathbb{R}$, which can be lifted to an action on the data for our classes (for properly chosen representatives $f : B \to \Omega SU(n)$).

It can be shown that qc_k stabilize to classes in the cohomology ring

$$H^*(\lim_{g,n\mapsto\infty} \mathcal{I}_{g,n}, \widehat{\mathrm{QH}}(\mathbb{CP}^\infty)). \tag{1.6}$$

Stabilization in n is analogous to to what is done in this paper, while stabilization in g, uses semi-simplicity of the quantum homology algebra, as in Teleman [15]. The hypothesis is then that qc_k are algebraically independent in this cohomology ring.

To put this conjecture into some perspective, consider the embedding

$$i : \mathcal{M}_{g,1} \subset \mathcal{I}_{g,n},$$

where $\mathcal{M}_{g,1}$ is the moduli space of closed genus g Riemann surfaces with one marked point. The map i is defined by sending $[\Sigma_g, \alpha, j] \in \mathcal{M}_{g,1}$ to the class of the triple $(\Sigma_g \times \mathbb{C}^n, \alpha, j)$, with $\Sigma_g \times \mathbb{C}^n$ endowed with a product holomorphic structure. On $\mathcal{M}_{g,1}$ there are Miller-Morita-Mumford characteristic classes c_k^m and Mumford's original conjecture was that they generate rational cohomology of the direct limit

$$\lim_{g\mapsto\infty} \mathcal{M}_{g,1},$$

where the limit is over natural (up to homotopy) inclusions $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$. This conjecture was proved by Madsen-Weiss as a consequence of a considerably stronger statement, [5].

On the other hand we have induced classes $qc_k \in H^*(\mathcal{M}_{g,1}, QH(\mathbb{CP}^{n-1}))$. Since i^* is surjective on cohomology, as we have a continous projection $pr : \mathcal{I}_{g,n} \to \mathcal{M}_{g,1}$, with $pr \circ i = id$) a special case of the above conjecture is that these classes are algebraically independent in the cohomology ring

$$H^*(\lim_{g\mapsto\infty} \mathcal{M}_{g,1}, \widehat{\mathrm{QH}}(\mathbb{CP}^\infty)),$$

For Miller-Morita-Mumford classes the corresponding statement is due to Morita. We may of course also conjecture that qc_k generate $H^*(\lim_{g\mapsto\infty} \mathcal{M}_{g,1}, \widehat{\mathrm{QH}}(\mathbb{CP}^\infty))$, which is an interesting "quantum" analogue of the Mumford conjecture.

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2. Setup

This section discusses all relevant constructions, and further outlines the main arguments.

2.1. **Quantum homology.** In definition of quantum homology $QH(M)$, or various Floer homologies one often uses some kind of Novikov coefficients with which $QH(M)$ has special grading making quantum multiplication graded. This is often done even for monotone symplectic manifolds (M, ω), where it is certainly technically unnecessary, at least if one is not concerned with grading. Here we choose to work over $\mathbb{C}$, which will also make definition of quantum classes more elegant and physical, as well as mathematically interesting. It also becomes more natural when we come to $QH(\mathbb{CP}^\infty)$, as that no longer has any natural grading.

Definition 2.1. *For a symplectic manifold (M, ω) we set $QH(M) = H_*(M, \mathbb{C})$, which we think of as ungraded vector space, and hence drop the subscript $*$.*

2.2. **Quantum product.** For integral generators $a, b \in H_*(M)$, this is the product defined by

$$a * b = \sum_{A \in H_2(M)} b_A e^{-i\omega(A)}, \tag{2.1}$$

where b_A is the homology class of the evaluation pseudocycle from the pointed moduli space, of J-holomorphic A-curves intersecting generic pseudocycles representing a, b, for a generic ω tamed J. This sum is finite in the monotone case: $\omega = kc_1(TM)$, with $k > 0$. The product is then extended to $QH(M)$ by linearity. For more technical details see [6].

2.3. **Quantum classes.** We now give a brief overview of the construction of classes

$$qc_k \in H^k(\Omega \text{Ham}(M, \omega), QH(M)),$$

for a monotone symplectic manifold (M, ω), originally defined in [11]. The reader may note that this construction extends Seidel representation, [14]. Since our coefficients have no torsion these classes are specified by a map of Abelian groups

$$\Psi : H_*(\Omega \text{Ham}(M, \omega), \mathbb{Q}) \to QH(M),$$

by setting

$$qc_k([f]) = \Psi([f]),$$

for $f : B \to \Omega \text{Ham}(M, \omega)$ a cycle. However, purely with this point of view we lose the extra structure of the Whitney sum formula. We now describe Ψ. By Smale's theorem rational homology is generated by cycles: $f : B^k \to \Omega \text{Ham}(M, \omega)$, where B is a closed oriented smooth manifold, which we may also assume to be smooth: the associated map $f : B \times S^1 \to \text{Ham}(M, \omega)$ is smooth). In fact, since ΩG has the homotopy type of a double loop space for any topological group G, by Milnor-Moore, Cartan-Serre [8], [1] the rational homology of $\Omega \text{Ham}(M, \omega)$ is *freely* generated via Pontryagin product by rational homotopy groups. In particular, the relations between cycles in homology can also be represented by smooth (in fact cylindrical) cobordisms.

Given this we may construct a bundle

$$p : P_f \to B,$$

with

$$P_f = B \times M \times D_0^2 \bigcup B \times M \times D_\infty^2 / \sim, \tag{2.2}$$

where $(b, x, 1, \theta)_0 \sim (b, f_{b,\theta}(x), 1, \theta)_\infty$, using the polar coordinates $(r, 2\pi\theta)$. With fiber modelled by a Hamiltonian fibration $M \hookrightarrow X \xrightarrow{\pi} S^2$.

Fix a family $\{j_{b,z}\}$, $b \in B$, $z \in S^2$ of almost complex structures on $M \hookrightarrow P_f \to B \times S^2$ fiber-wise compatible with ω.

Definition 2.2. *A family $\{J_b\}$ is called* **π-compatible** *if:*

- *The natural map $\pi : (X_b, J_b) \to (S^2, j)$ is J_b holomorphic for each b.*
- *J_b preserves the vertical tangent bundle of $M \hookrightarrow X_b \to S^2$ and restricts to the family $\{j_{b,z}\}$.*

The importance of this condition is that it forces bubbling to happen in the fibers, where it is controlled by monotonicity of (M, ω).

The map Ψ we now define measures part of the degree of quantum self intersection of a natural submanifold $B \times M \subset P_f$. The entire quantum self intersection is captured by the total quantum class of P_f. We define Ψ as follows:

$$\Psi([B, f]) = \sum_{\widetilde{A} \in j_*(H_2^{sect}(X))} b_{\widetilde{A}} \cdot e^{-i\mathcal{C}(\widetilde{A})}, \tag{2.3}$$

Here,

- $H_2^{sect}(X)$ denotes the section homology classes of X.
- $\mathcal{C}$ is the coupling class of Hamiltonian fibration
$$M \hookrightarrow P_f \to B \times S^2, \text{ see [3, Section 3]}. \tag{2.4}$$
Restriction of $\mathcal{C}$ to the fibers $X \subset P_f$ is uniquely determined by the condition
$$i^*(\mathcal{C}) = [\omega], \qquad \int_M \mathcal{C}^{n+1} = 0 \in H^2(S^2). \tag{2.5}$$
where $i : M \to X$ is the inclusion of fiber map, and the integral above denotes the integration along the fiber map for the fibration $\pi : X \to S^2$.
- The map $j_* : H_2^{sect}(X) \to H_2(P_f)$ is induced by inclusion of fiber.
- The coefficient $b_{\widetilde{A}} \in H_*(M)$ is defined by duality:
$$b_{\widetilde{A}} \cdot_M c = ev_0 \cdot_{B \times M} [B] \otimes c,$$
where
$$ev_0 : \mathcal{M}(P_h, \widetilde{A}, \{J_b\}) \to B \times M$$
$$ev_0(u, b) = (u(0), b)$$
denotes the evaluation map from the space
$$\mathcal{M}(P_f, \widetilde{A}, \{J_b\}) \tag{2.6}$$
of tuples (u, b), u is a J_b-holomorphic section of X_b in class $\widetilde{A}$ and $\cdot_M, \cdot_{B \times M}$ denote the intersection pairings.
- The family $\{J_b\}$ is π-compatible in the sense above.

The fact that Ψ is well defined with respect to various choices: the representative $[f]$, and the family $\{J_b\}$, is described in more detail in [11], however this is a very standard argument in Gromov-Witten theory: a homotopy of this data gives rise to a cobordism of the above moduli spaces.

Also the sum in (2.3), is actually finite. If $T^{vert}P_f$ denotes the vertical tangent bundle of (2.4), then the natural restrictions on the dimension of the moduli space

$$2n + 2k + 2\langle c_1(T^{vert}P_f), \widetilde{A}\rangle, \tag{2.7}$$

give rise to bounds on $\langle c_1(T^{vert}P_f), \widetilde{A}\rangle$, i.e. to degree d, where $\widetilde{A} = [S^2] + d[line]$, as a class in $X = M \times S^2$. Consequently only finitely many such classes can contribute.

Bundles of the type P_f above have a Whitney sum operation. Given P_{f_1}, P_{f_2} we get the bundle $P_{f_1} \oplus P_{f_2} \equiv P_{f_2 \cdot f_1}$, where $f_2 \cdot f_1$ is the pointwise multiplication of the maps $f_1, f_2 : B \to \Omega\text{Ham}(M, \omega)$, using the natural topological group structure of $\Omega\text{Ham}(M, \omega)$. Geometrically this corresponds to doing connected sum on the fibers [11][Section 4.4] (which themselves are fibrations over S^2), and the set of (suitable) isomorphism classes of such bundles over B form an Abelian group $\mathcal{P}_B$, the group of homotopy classes of

$$f : B \to \Omega\text{Ham}(M, \omega).$$

For $P_f \in \mathcal{P}_B$ we set

$$qc_k(P_f) = f^* qc_k. \tag{2.8}$$

We may now state the properties satisfied by these classes, which we call axioms even though they do not characterize, these are verified in [11]. *Quantum classes* are a sequence of functions

$$qc_k : \mathcal{P}_B \to H^k(B, QH(M))$$

satisfying the following axioms:

Axiom 1 (Naturality)**.** *For a map $g : B_1 \to B_2$:*

$$g^* qc_k(P_2) = qc_k g^*(P_2).$$

Axiom 2 (Whitney sum formula)**.** *If $P, P_1, P_2 \in \mathcal{P}_B$ and $P = P_1 \oplus P_2$, then*

$$qc(P) = qc(P_1) \cup qc(P_2),$$

where $\cup$ is the cup product of cohomology classes with coefficients in the quantum homology ring $QH(M)$ and $qc(P)$ is the total characteristic class

$$qc(P) = qc_0(P) + \ldots + qc_m(P), \tag{2.9}$$

where m is the dimension of B.

Axiom 3 (Partial normalization)**.** $\langle qc_0(P), [pt]\rangle = \boldsymbol{1} = [M] \in QH(M)$.

Corollary 2.3. Ψ *is a ring homomorphism from $\Omega Ham(M, \omega)$ with its Pontryagin product to $QH(M)$ with its quantum product.*

Let us now specialize to $M = \mathbb{CP}^{n-1}$ with its natural symplectic form ω normalized by $\omega(A) = 1$, for A the class of the line. And let us restrict our attention to the subgroup $\Omega SU(n) \subset \Omega\text{Ham}(\mathbb{CP}^n)$. The map $\Psi : \Omega SU(n) \to QH(\mathbb{CP}^{n-1})$ will be denoted by Ψ^n. Note that in this case

$$\Psi^n([pt]) = \mathbf{1} = [\mathbb{CP}^{n-1}], \tag{2.10}$$

since $SU(n)$ is simply connected. We will proceed to induce Ψ on the limit ΩSU. This will allow us to arrive at a true normalization axiom, completing the axioms in that setting, and will also allow us to make contact with the splitting principle on BU. Let

$$i : \Omega SU(n) \to \Omega SU(m),$$

$$i(\gamma)(\theta) = \begin{pmatrix} \gamma(\theta) & & & \\ & 1 & & \\ & & \ldots & \\ & & & 1 \end{pmatrix}$$

for $m > n$, and

$$j : \mathbb{CP}^{n-1} \to \mathbb{CP}^{m-1},$$
$$j([z_0, \ldots, z_n]) = [z_0, \ldots, z_n, 0, \ldots, 0]$$

be compatible inclusions. For $a, b \in H_*(\mathbb{CP}^{n-1})$, of pure degree

$$j_*(a * b) = j_*(a) * j_*(b)$$

if and only if $a*b$ has a quantum correction, or if degree of $\deg a + \deg b + 2 \leq 2(n-1)$. In particular the direct limit of abelian groups

$$\lim_n QH(\mathbb{CP}^{n-1})$$

makes sense as a ring which we denote by $QH(\mathbb{CP}^\infty)$, i.e. it's the natural formal quantum ring on $H_*(\mathbb{CP}^\infty, \mathbb{C})$. Denote by $\widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$ its unital completion, and $\mathbf{1}$ its unit, which we may informally view as $[\mathbb{CP}^\infty]$. This ring is just a free polynomical algebra over $\mathbb{C}$ on one generator.

Here is the main step.

Proposition 2.4. *For a cycle* $f : B^{2k} \to \Omega SU(n)$

$$\Psi^m([i \circ f]) = j_* \Psi^n([f]), \tag{2.11}$$

for $2k$ *in stable range* $[2, 2n-2]$.

The above proposition is indeed fairly surprising. For even if one believes that the bundles induced by $j \circ f$, f have basically the "same twisting", the ambient spaces (fibrations, fibers) are certainly different and this same twisting can manifest itself in different looking elements of quantum homology. And in fact this different manifestation is certain to happen unless $\Psi^n([f])$ is of the form $c \cdot e^i \in QH_*(\mathbb{CP}^{n-1})$, i.e. unless the only contribution to $\Psi^n([f])$ is coming from the section class $\widetilde{A} \in H_2(\mathbb{CP}^{n-1} \times S^2)$ with $\mathcal{C}(\widetilde{A}) = \omega(\widetilde{A}) = -1$. For if its not, the above equality is impossible by dimension considerations, (this is a straightforward exercise involving definition of Ψ.)

Corollary 2.5. *There is an induced map* $\Psi : BU \simeq \Omega SU \to \widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$*, induced by* (2.11) *by postulating that* $\Psi([pt]) = \mathbf{1}$ *(cf.* (2.10)*) and consequently cohomology classes*

$$qc_k^\infty \in H^{2k}(BU, \widehat{QH}(\mathbb{CP}^\infty))$$
$$qc_k^\infty([f]) = \Psi([f]),$$

for $f : B^{2k} \to BU$ *a cycle.*

Theorem 2.6. *Let* $\widetilde{K}(B)$ *denote the reduced K-theory group of* B*. The classes* $qc_k^\infty \in H^{2k}(BU, \widehat{\mathrm{QH}}(\mathbb{CP}^\infty))$ *satisfy and are determined by the following axioms:*

Axiom 4 (Functoriality)**.** *For a map* $g : B_1 \to B_2$*, and* $P_2 \in \widetilde{K}(B)$*:*

$$g^* qc_k^\infty(P_2) = qc_k^\infty(g^* P_2).$$

Axiom 5 (Whitney sum formula)**.** *If* $P, P_1, P_2 \in \widetilde{K}(B)$ *and* $P = P_1 \oplus P_2$*, then*

$$qc^\infty(P) = qc^\infty(P_1) \cup qc^\infty(P_2),$$

where $\cup$ is the cup product of cohomology classes with coefficients in the ring $\widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$ and $qc^\infty(P)$ is the total characteristic class

$$qc^\infty(P) = \mathbf{1} + \ldots + qc^\infty_m(P), \tag{2.12}$$

where m is the dimension of B.

Moreover, we will prove in Section 3 that they satisfy the following:

Axiom 6 (Normalization)**.** *If $f_l : \mathbb{CP}^k \to BU$ is the classifying map for the stabilized canonical line bundle, then*

$$\Psi(f_l]) = [\mathbb{CP}^{k-1}]e^i. \tag{2.13}$$

Verification of the first two axioms is immediate from the corresponding axioms of the classes $qc_k \in H^{2k}(\Omega SU(n), QH(\mathbb{CP}^{n-1}))$, using the well known fact that under the Bott isomorphism $BU \simeq \Omega SU$, the group $\widetilde{K}(B)$ corresponds to the group of homotopy classes of maps $f : B \to \Omega SU$. Verification of the last axiom is done in Section 3.

As we see the classes qc^∞_k formally have the same axioms as Chern classes, but they do not have the dimension property, i.e. $qc_k(P)$ does not need to vanish if P is stabilization of rank r bundle with $r < 2k$: this already fails for the canonical line bundle.

Proof of Theorem 1.1. To show generation, note that by the splitting principle, it is enough to show that qc classes generate cohomology of $BT \subset BU$, where T is the infinite torus. We need to show that for every $f : B^{2k} \to BT$ non vanishing in homology some quantum Chern number of P_f is non-vanishing. That is we must show:

$$\langle \prod_i qc^{\alpha_i}_{\beta_i}(P_f), [B]\rangle \neq 0, \text{ for some } \beta_i, \alpha_i, \text{ s.t. } \sum_i 2\beta_i \cdot \alpha_i = 2k. \tag{2.14}$$

Clearly we may assume that $B^{2k} = \prod_i [\mathbb{CP}^{\alpha(i)}]$ and $f = \widetilde{f}^l_1 \cdot \ldots \widetilde{f}^l_i \cdot \ldots$, where $\sum_i \alpha(i) = k$, and $\widetilde{f}^l_i : \prod_i [\mathbb{CP}^{\alpha(i)}] \to BT$ is the classifying map for the pullback of the canonical line bundles on $\mathbb{CP}^{\alpha(i)}$ under the projection $\prod_i [\mathbb{CP}^{\alpha(i)}] \to \mathbb{CP}^{\alpha(i)}$. But then the claim follows by Axioms 6, and 5.

To show algebraic independence, note that SU has non-vanishing rational homotopy groups in each odd degree, indeed the rational homotopy type of SU is well known to be

$$S^3 \times S^5 \times \ldots \times S^{2k-1} \ldots.$$

Consequently, $BU \simeq \Omega SU$ has non-vanishing rational homotopy groups in each even degree, and by Milnor-Moore theorem these spherical generators do not vanish in homology. (In this case Milnor-Moore, Cartan-Serre theorem says that $H_*(\Omega SU, \mathbb{Q})$ is freely generated by rational homotopy groups via Pontryagin product, see [8], [1]). For such a spherical homology class in degree $2k$, clearly only qc_k may be non-vanishing, and must be non-vanishing since qc_k generate. ☐

Remark/Question 2.7. *The space ΩSU is actually a homotopy ring space. On BU the ring multiplication corresponds to tensor product of vector bundles. In terms of bundles P_f over B there is a kind of tensor product in addition to Whitney sum. The tensor product is induced by the Abelian multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which induces a multiplication on fibrations*

$$\pi : \mathbb{CP}^\infty \hookrightarrow X \to S^2.$$

One then needs to check that $P_{f_1} \otimes P_{f_2} \in \mathcal{P}_B$. It is then likely that this is the same as the ring coming from vector bundle tensor product. Or perhaps its a different ring structure? In any case, one would have to check that the proposed tensor product is well behaved for quantum classes, analogously to the case of Chern characters. If this ring is indeed different, then this is where "exoticity" of quantum classes may have an interesting expression.

3. Proofs

We are going to give two proofs of Proposition 2.4. The first one is more difficult, but less transcendental of the two, and we present here because it has the advantage of suggesting a route to computation of quantum classes in the non-stable range. Moreover, the energy flow picture in this argument is necessary for proof of Theorem 1.4. Unfortunately, this first proof requires some basics of the theory of loop groups. The second proof was suggested to me by Dusa McDuff, and basically just uses the index theorem and automatic transversality. For convenience, these will be presented independently of each other, so can be read in any order.

Proof of Proposition 2.4. We will need the following.

Proposition 3.1. [9, Proposition 8.8.1] *There are natural maps $i^n_B : \Omega SU(n) \to BU$ inducing isomorphism of homotopy groups up to dimension $2n - 2$.*

The proof of this fact uses cellular stratification induced by the energy functional E on $\Omega SU(n)$. For various details in the following the reader is referred to [9, Sections 8.8, 8.9]. For us, the most important aspect of the proof of the proposition above is that there is a holomorphic cell decomposition of Kahler manifold $\Omega SU(n)$ up to dimension $2n-2$, with cells indexed by homomorphisms $\lambda : S^1 \to T$ with all weights either $1, -1, 0$, where $T \subset SU(n)$ is a fixed maximal torus. The closure $\bar{C}_\lambda$ of each such cell C_λ, is the closure of unstable manifold (for E) of a (certain) cycle in the E level set of λ. Let

$$f_\lambda : B^{2k}_\lambda \to \Omega SU(n)$$

be the compactified $2k$-pseudocycle representing $\bar{C}_\lambda$.

Each $\gamma \in \bar{C}_\lambda$ is a polynomial loop, i.e. extends to a holomorphic map $\gamma_{\mathbb{C}} : \mathbb{C}^\times \to SL_{\mathbb{C}}(n)$. Consequently, X_γ has a natural holomorphic structure, so that the complex structure on each fiber $M_z \to X_\gamma \to S^2$ is tamed by ω, and so we have a natural admissible family of complex structures on P_{f_λ}. In fact X_γ is Kahler but this will not be important to us. What will be important is that X_γ is biholomorphic to X_{γ_∞}, where $\gamma_\infty : S^1 \to SU(n)$, $\gamma_\infty \in \bar{C}_\lambda$ is S^1 subgroup, which is a limit in forward time of the negative gradient flow trajectory of γ. In particular γ_∞ has all weights either $1, -1, 0$. For $\gamma = f_\lambda(b)$ we call the corresponding γ_∞

$$\gamma^b_\infty. \tag{3.1}$$

Aside from the condition on weights of the circle subgroups, here is another point where the stability condition $2k \leq 2n-2$ comes in. Let σ_{const} denote some constant section of $X \simeq \mathbb{CP}^{n-1} \times S^2$, which is our name for the topological model of the fibers of P_f. For a class

$$\widetilde{A} = [\sigma_{const}] - d \cdot [line] \in H_2(X)$$

the expected dimension of $\mathcal{M}(P_f, \widetilde{A}, \{J_b\})$ (see (2.6)) is

$$2n + 2k + 2c_1^{vert}(\widetilde{A}) \leq 2n + 2n - 2 - 2d \cdot n < 0 \text{ unless } d \leq 1,$$

where c_1^{vert} is first Chern class of the vertical tangent bundle of

$$M \hookrightarrow P_{f_\lambda} \to B_\lambda \times S^2.$$

Thus, $\widetilde{A}$ can contribute to $\Psi(P_{f_\lambda})$ only if $d = 1$, ($d = 0$ can only contribute to $c_0^q(P_f)$; which can be checked by analyzing the definition.) This fact will be crucial in the course of this proof. We set

$$S = [\sigma_{const}] - A \in H_2(X).$$

Let us first understand moduli spaces of holomorphic S curves in X_{γ_∞}, with γ_∞ an S^1 subgroup of the type above. Each X_{γ_∞} is biholomorphic to $S^3 \times_{\gamma_\infty} \mathbb{CP}^{n-1}$ i.e. the space of equivalence classes of tuples

$$[z_1, z_2, x], \quad x \in \mathbb{CP}^{n-1}, \tag{3.2}$$

under the action of S^1

$$e^{2\pi it} \cdot [z_1, z_2, x] = [e^{-2\pi it} z_1, e^{-2\pi it} z_2, \gamma_\infty(t)x], \tag{3.3}$$

using complex coordinates z_1, z_2 on S^3.

Let H_{γ_∞} be the normalized generating function for the action of γ_∞ on $\mathbb{CP}^{n-1}$. Each x in the max level set $F_{\max}$ of H_{γ_∞}, gives rise to a section $\sigma_x = S^3 \times_{\gamma_\infty} \{x\}$ of X_{γ_∞}. It can be easily checked that $[\sigma_x] = S$. Moreover, by elementary energy considerations it can be shown that these are the only stable holomorphic S sections in X_{γ_∞}, see discussion following Definition 3.3. in [12]. Consequently the moduli space $\mathcal{M} = \mathcal{M}(P_{f_\lambda}, S, \{J_b\})$ is compact. It's restriction over open negative gradient trajectories $\mathbb{R} \to \widetilde{C}_\lambda$ asymptotic to γ_∞ in forward time is identified with $\mathbb{R} \times F_{\max}$, where $F_{\max}$ is as above.

The regularized moduli space can be constructed from $\mathcal{M}$ together with kernel, cokernel data for the Cauchy Riemann operator for sections $u \in \mathcal{M}$. This is well understood by know, see for example an algebro-geometric approach in [4]. It is however interesting that our moduli space is highly singular.

What we need to show is that $\mathcal{M}$ and the local data for the Cauchy Riemann operator are identified with $\mathcal{M}^s = \mathcal{M}(P_{i \circ f_\lambda}, S, \{J_b\})$ and the corresponding local data for the Cauchy Riemann operator, where $i \circ f_\lambda : B_\lambda \to \Omega SU(m)$. The fact that the moduli spaces $\mathcal{M}$, $\mathcal{M}^s$ are identified follows more or less immediately from the preceding discussion. The inclusion map $i : \Omega SU(n) \to \Omega SU(m)$ takes gradient trajectories in $\Omega SU(n)$ to gradient trajectories in $\Omega SU(m)$ and any circle subgroup is taken to a circle subgroup with $m - n$ new 0-weights. There is a natural map $j : P_f \to P_{i \circ f_\lambda}$, and its clear that it identifies $\mathcal{M}$ with $\mathcal{M}^s$. We will denote $j(X_b)$ by $X^s_{j(b)}$, and an element in $\mathcal{M}^s$ identified to an element $u = (\sigma_x, b) \in \mathcal{M}$ by $j(u)$, where $b \in B_{f_\lambda}$.

We show that the linearized Cauchy Riemann operators at u, $j(u)$ have the same kernel and cokernel. Let V denote the infinite dimensional domain for the linearized CR operator D_u at u. There is a subtlety here, since our target space P_{f_λ} is a smooth stratified space we actually have to work a strata at a time, but we suppress this. The domain for the CR operator $D_{j(u)}$ is the appropriate Sobolev completion of the space of C^∞ sections of the bundle $j(u)^* T^{vert} X^s_{j(b)}$, where

$$j(u)^* T^{vert} X^s_{j(b)} \simeq S^3 \times_{\gamma_\infty^{j(b)}} \mathbb{CP}^{m-1},$$

is a holomorphic vector bundle, and $\simeq$ is isomorphism of holomorphic vector bundles. Similarly,

$$u^*T^{vert}X_b \simeq S^3 \times_{\gamma^b_\infty} \mathbb{CP}^{n-1}.$$

Consequently $j(u)^*T^{vert}X^s_{j(b)}$ holomorphically splits into line bundles with Chern numbers either $0, -2, -1$. All the Chern number 0 and -2 summands are identified with corresponding summands of $u^*T^{vert}X_b$, and consequently the domain of $D_{j(u)}$ in comparison to D_u is enlarged by the space of sections W, of sum of Chern number -1 holomorphic line bundles. But in our setting $D_{j(u)}$ coincides with Dolbeault operator and so we will have no "new kernel or cokernel" in comparison to D_u. More precisely:

$$D_{j(u)} : W \to \Omega^{0,1}(S^2, j(u)^*T^{vert}X^s_{j(b)})/\Omega^{0,1}(S^2, u^*T^{vert}X)$$

is an isomorphism, which concludes our argument. $\square$

Second proof of Proposition 2.4. It is enough to verify the stabilization property for $m = n+1$. We have to again make use of the homological stability condition, as in the first proof. Let $f : B \to \Omega SU(n)$ be a map of a smooth, closed oriented manifold. For a class

$$\widetilde{A} = [\sigma_{const}] - d \cdot [line] \in H_2(X)$$

the expected dimension of $\mathcal{M}(P_f, \widetilde{A}, \{J_b\})$ (see (2.6)) is

$$2n + 2k + 2c_1^{vert}(\widetilde{A}) \leq 2n + 2n - 2 - 2d \cdot n < 0 \text{ unless } d \leq 1,$$

where c_1^{vert} is first Chern class of the vertical tangent bundle of

$$M \hookrightarrow P_f \to B \times S^2.$$

Thus, $\widetilde{A}$ can contribute to $\Psi(P_f)$ only if $d = 1$, ($d = 0$ can only contribute to $c_0^q(P_f)$; which can be checked from the definition.) We set $S = [\sigma_{const}] - [line]$.

Let $j : \Omega SU(n) \to \Omega SU(n+1)$ be the inclusion map, and denote $j \circ f$ by f'. Take a family of almost complex structures $\{J_{f,b}\}$ on P_f for which the moduli space $\mathcal{M}(P_f, S, \{J_{f,b}\})$ is regular. Extend $\{J_{f,b}\}$ to a family $\{J_{f',b}\}$ on $P_{f'}$ in any way. Consequently, for the families of almost complex structure $\{J_{f,b}\}$ on P_f and $\{J_{f',b}\}$ on $P_{f'}$ the natural embedding of P_f into $P_{f'}$ is holomorphic. The intersection number of a curve $u \in \mathcal{M}(P_{f'}, S, \{J_{f',b}\})$ with $P_f \subset P_{f'}$, is c_1 of the normal bundle of $\mathbb{CP}^{n-1}$ inside $\mathbb{CP}^n$ evaluated on $-[line]$, i.e. -1. Consequently by positivity of intersections, (see [6]) for the family $\{J_b\}$ all the elements of the space $\mathcal{M}(P_{f'}, S, \{J_b\})$ are contained inside the image of embedding of P_f into $P_{f'}$. We now show that $\{J_{f',b}\}$ is also regular. This will immediately yield our proposition. The pullback of the normal bundle to embedding, by $u_b \in \mathcal{M}(P_{f'}, S, \{J_{f',b}\})$ is $\mathcal{O}(-1)$. So we have an exact sequence

$$u^*T^{vert}P_f \to (j \circ u)^*T^{vert}P_{f'} \to \mathcal{O}(-1). \tag{3.4}$$

By construction of $\{J^{pert}_{f',b}\}$ the real linear CR operator $D_{i \circ u}$ is compatible with this exact sequence. More explicitly, we have the real linear CR operator

$$\Omega^0(S^2, \mathcal{O}^{-1}) \to \Omega^{0,1}(S^2, \mathcal{O}(-1)) \tag{3.5}$$

induced by $D_{j \circ u_b}$ on

$$\Omega^0(S^2, (j \circ u)^*T^{vert}P_{f'}/u^*T^{vert}P_f),$$

with image

$$\Omega^{0,1}(S^2, (j \circ u)^* T^{vert} P_{f'}/u^* T^{vert} P_f),$$

since $u^* T^{vert} P_f \subset (j \circ u)^* T^{vert} P_{f'}$ is $J_{f',b}$ invariant. Such an operator is surjective by the Riemann-Roch theorem, [6]. Consequently $D_{j \circ u_b}$ is surjective. □

(Verification of Axiom 6). For the purpose of this computation we will change our perspective somewhat, since the moduli spaces at which we arrived in the first proof of Proposition 2.4 were very singular. (Although, as hinted in the Introduction we maybe able to also use virtual localization, and stay within the setup of that argument.) The reader may wonder if the perspective in the following argument could also be used to attack Proposition 2.4 as well, but this appears to be impossible as the picture below doesn't stabilize very nicely.

Consider the path space from I to $-I$ in $SU(2n)$: $\Omega_{I,-I} SU(2n)$. Fixing a path from I to $-I$, say along a minimal geodesic

$$p = \begin{pmatrix} e^{\pi i t} & & & & & \\ & \cdots & & & & \\ & & e^{\pi i \theta} & & & \\ & & & e^{-\pi i t} & & \\ & & & & \cdots & \\ & & & & & e^{-\pi i t} \end{pmatrix}, \tag{3.6}$$

there is an obvious map $m : \Omega_{I,-I} SU(2n) \to \Omega SU(2n)$. We will make use of the Bott map.

Theorem 3.2 (Bott). *There is a natural inclusion $m \circ i_B^n : Gr_n(2n) \to \Omega SU(2n)$, inducing an isomorphism on homotopy (homology) groups in dimension up to $2n$.*

The map i_B^n is the inclusion of the manifold of minimal geodesics in $SU(2n)$ from I to $-I$, into $\Omega_{I,-I} SU(2n)$. Any such geodesic is conjugate to p above and is determined by the choice of n complex dimensional $e^{\pi i \theta}$ weight space in $\mathbb{C}^{2n}$.

The bound $2n$ is due to the fact that the index of a non-minimal closed geodesic from I to $-I$ in $SU(2n)$ is more than $2n$. For more details on Bott map and this argument we refer the reader to the proof of Bott periodicity for the unitary group in [7].

Let $f_l : \mathbb{CP}^k \to BU$ be as in wording of the axiom. For $2n$ sufficiently large the homotopy class of $i_B \circ f_l$ is represented by

$$i \circ m \circ i_B^n \circ \widetilde{f} : \mathbb{CP}^k \to BU,$$

where

$$\widetilde{f} : \mathbb{CP}^k \to Gr_n(2n),$$

is the map

$$\widetilde{f}(l) = pl^{n-1} \oplus l, \tag{3.7}$$

where pl^{n-1} is a fixed complex $n-1$ subspace in $\mathbb{C}^{2n}$, l denotes a 1 complex dimensional subspace in the orthogonal $\mathbb{C}^{n+1}$, and where i is just the inclusion $\Omega SU(2n) \to \Omega SU$, however i is suppressed from now on. Therefore

$$i_B^n \circ \widetilde{f}(l) = \begin{pmatrix} e^{\pi i t} & & & \\ & \cdots & & \\ & & e^{\pi i t} & \\ & & & \gamma_l, \end{pmatrix} \tag{3.8}$$

where the top $n-1$ by $n-1$ block is the diagonal matrix with entries $e^{\pi it}$, and the bottom block $\gamma_l \in SU(n+1)$ has $e^{\pi it}$ weight space determined by $l \subset \mathbb{C}^{n+1}$.

We can tautologically express $m \circ i^n_B \circ \widetilde{f}$ as the composition of the map

$$g = i^n_B \circ \widetilde{f} : \mathbb{CP}^n \to \Omega PSU(2n),$$

and $L : \Omega PSU(2n) \to \Omega SU(2n)$, with L defined as follows: for a loop $\gamma : S^1 \to PSU(2n)$ pointwise multiply with the loop $p : S^1 \to PSU(2n)$ and lift to the universal cover $SU(2n)$. Since $\gamma \cdot p$ is contractible the lift is a closed loop. By Corollary 2.3

$$\Psi(m \circ i^n_B \circ \widetilde{f}) = \Psi(g) * \Psi(p), \tag{3.9}$$

where $*$ is the quantum product and we are in-distinguishing notation for the map $p : S^1 \to PSU(2n)$ and associated map $p : [pt] \to \Omega PSU(2n)$. Note that the restriction of Ψ to $H_0(\Omega\mathrm{Ham}(M, \omega), \mathbb{Q})$ is the *Seidel representation* map S, [14].

Lemma 3.3.

$$\Psi(p) = S(p) = [\mathbb{CP}^{n-1}]e^{i/2}.$$

Proof. Our bundle X_p over S^2 is naturally isomorphic to $S^3 \times_p \mathbb{CP}^{2n-1}$, and so has a natural holomorphic structure. Further, we have natural holomorphic sections $\sigma_x = S^3 \times_p \{x\}$, where $x \in F_{\max} \simeq \mathbb{CP}^{n-1}$ the max level set of the generating function H_p of the Hamiltonian action of p on $\mathbb{CP}^{2n-1}$. The class of such a section will be denoted $[\sigma_{\max}]$. It can be easily checked that $c_1^{vert}([\sigma_{\max}]) = -n$ and that $\mathcal{C}([\sigma_{\max}]) = -1/2$. By elementary consideration of energy it can be shown that the above are the only stable holomorphic $[\sigma_{\max}]$ class sections of X_p; see discussion following Definition 3.3, [12]. The corresponding moduli space is regular, (the weights of the holomorphic normal bundle to sections $\sigma_{\max}$ are 0 and -1). Further, from dimension conditions it readily follows that there can be no other contributions to $\Psi(p)$ and so $\Psi(p) = [\mathbb{CP}^{n-1}]e^{i/2}$. $\square$

Similarly, the bundle P_g has a natural admissible family $\{J_b\}$, with all fibers (X_b, J_b) biholomorphic to each other. Each X_b is identified with a naturaly complex manifold $S^3 \times_{g_b} \mathbb{CP}^{2n-1}$, where g_b denotes the circle subgroup $g(b)$ of $PSU(2n)$, cf (3.3). The previous discussion can be applied to each X_b. And it implies that $\mathcal{M} = \mathcal{M}(P_g, [\sigma_{\max}], \{J_b\})$ is compact and fibers over $B = \mathbb{CP}^k$ with fiber $\mathbb{CP}^{n-1}$. For

$$\widetilde{A} = [\sigma_{\max}] - d[line] \in H_2(X)$$

The virtual dimension of $\mathcal{M}(P_f, \widetilde{A}, \{J_b\})$ is

$$2n + (4n-2) + 2c_1^{vert}(\widetilde{A}) = 2n + (4n-2) - 2n - d4n < 0, \tag{3.10}$$

unless $d = 0$, i.e. $\widetilde{A} = [\sigma_{\max}]$ is the only class that can contribute. Thus, we only need to understand the contribution from $[\sigma_{\max}]$ class. Once again $\mathcal{M} = \mathcal{M}(P_g, [\sigma_{\max}], \{J_b\})$ is regular, i.e. the associated linearized Cauchy Riemann operator is onto for every $(\sigma_x, b) \in \mathcal{M}$, since by construction, the normal bundle

$$N = S^3 \times_{g_b} T_x\mathbb{CP}^{2n-1}$$

to a section $u \in \mathcal{M}$ is holomorphic with weights $0, -1$. By definition $\Psi(g)$ is determined by the intersection number of the evaluation cycle

$$e_0 : \mathcal{M} \to \mathbb{CP}^k \times \mathbb{CP}^{2n-1}, \tag{3.11}$$

$$e_0(u) = u(0) \tag{3.12}$$

with the cycle $[\mathbb{CP}^k] \otimes [\mathbb{CP}^{n-k}] \in H_*(\mathbb{CP}^k \times \mathbb{CP}^{2n-1})$. If we take a representative for $[\mathbb{CP}^{n-k}]$ not intersecting $\mathbb{CP}^{n-2} \subset \mathbb{CP}^{2n-1}$ corresponding under projectivization to the complex subspace pl^{n-1} (see (3.7)), then it is geometrically clear from construction that the two cycles intersect in a point by construction of the map g, and intersect transversally. It follows that $\Psi(g) = [\mathbb{CP}^{n-1+k}]e^{i/2}$. Therefore, $\Psi(i_B \circ f_l) = [\mathbb{CP}]^{n-1}e^{i/2} * [\mathbb{CP}^{n-1+k}]e^{i/2} = [\mathbb{CP}^{k-1}]e^i$, since in this case there is only classical contribution to the quantum intersection product. □

Proof of Theorem 1.4. Let $0 \neq a \in H_{2k}(\Omega SU(n))$, with $0 \leq 2k \leq 2n-2$. By Corollary 1.2 we have that

$$\langle \prod_i qc^{\alpha_i}_{\beta_i}, a\rangle \neq 0, \tag{3.13}$$

for some α_i, β_i. Since these classes are pull-backs of the classes $qc_k \in H_{2k}(\Omega \mathrm{Ham}(\mathbb{CP}^{n-1}, \omega))$, the same holds for these latter cohomology classes and the cycle i_*a.

Lemma 3.4. *For any representative $f : B \to \Omega Ham(\mathbb{CP}^{n-1}, \omega)$ for a,*

$$L^+(i \circ f(b)) \geq 1$$

for some $b \in B$.

Proof. This is essentially [12, Lemma 3.2], and we reproduce it's proof here for convenience. The total space of P_f is

$$P_f = B \times \mathbb{CP}^{n-1} \times D^2_0 \bigcup B \times \mathbb{CP}^{n-1} \times D^2_\infty / \sim, \tag{3.14}$$

where $(b, x, 1, \theta)_0 \sim (b, f_{b,\theta}(x), 1, \theta)_\infty$, using the polar coordinates $(r, 2\pi\theta)$. The fiberwise family of Hamiltonian connections $\{\mathcal{A}_b\}$ are induced by a family of certain closed forms $\{\widetilde{\Omega}_b\}$, which we now describe, by declaring horizontal subspaces of $\mathcal{A}_b$ to be $\widetilde{\Omega}_b$-orthogonal to the vertical subspaces of $\pi : F_b \to S^2$, where F_b is the fiber of P_f over $b \in S^k$.

The construction of this family mirrors the construction in Section 3.2 of [11]. First we define a family of forms $\{\widetilde{\Omega}_b\}$ on $B \times \mathbb{CP}^{n-1} \times D^2_\infty$.

$$\widetilde{\Omega}_b|_{D^2_\infty}(x, r, \theta) = \omega - d(\eta(r)H^b_\theta(x)) \wedge d\theta \tag{3.15}$$

Here, H^b_θ is the generating Hamiltonian for $f(b)$, normalized so that

$$\int_M H^b_\theta \omega^n = 0,$$

for all θ and the function $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(r),$$

and

$$\eta(r) = \begin{cases} 1 & \text{if } 1-\delta \leq r \leq 1, \\ r^2 & \text{if } r \leq 1-2\delta, \end{cases}$$

for a small $\delta > 0$.

It is not hard to check that the gluing relation $\sim$ pulls back the form $\widetilde{\Omega}_b|_{D^2_\infty}$ to the form ω on the boundary $\mathbb{CP}^{n-1} \times \partial D^2_0$, which we may then extend to ω on the whole of $\mathbb{CP}^{n-1} \times D^2_0$. Let $\{\widetilde{\Omega}_b\}$ denote the resulting family on X_b. The forms $\widetilde{\Omega}_b$ on F_b restrict to ω on the fibers $\mathbb{CP}^{n-1} \hookrightarrow F_b \to S^2$ and the 2-form obtained by fiber-integration $\int_{\mathbb{CP}^{n-1}} (\widetilde{\Omega}_b)^{n+1}$ vanishes on S^2. Such forms are called *coupling forms*, which is a notion due to Guillemin, Lerman and Sternberg [2]. We then have a family of closed forms

$$\Omega_b = \widetilde{\Omega}_b + \max_{x \in \mathbb{CP}^{n-1}} H^b_\theta \, d\eta \wedge d\theta. \tag{3.16}$$

A $J^{\mathcal{A}}_b$-holomorphic section of F_b in class S gives rise to a lower bound

$$1 = -\langle [\widetilde{\Omega}_b], S \rangle \leq \int_{S^2} \max_{x \in \mathbb{CP}^{n-1}} H^b_\theta \, d\eta \wedge d\theta = L^+(f(b)). \tag{3.17}$$

□

Of course such a section must exist by the discussion following (3.13). The theorem follows once we note that there is a representative $f' : B \to \Omega \mathrm{Ham}(\mathbb{CP}^{n-1}, \omega)$ for such an a, in the form $i \circ f$, with $f : B \to \Omega SU(n)$, s.t. the image of $f' = i \circ f$ is contained in the sublevel set $\Omega^1 \mathrm{Ham}(\mathbb{CP}^{n-1} \omega)$. Such a representative is readily found from the energy flow stratification picture of $\Omega SU(n)$, and we have already used it, see the proof of Proposition 2.4. □

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